

# Partial zeta functions

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## Abstract

In this paper, we study analytic properties of zeta functions defined by partial Euler products.

## 1 Introduction

In the previous paper [HW], we have studied the splitting densities of prime geodesics of negatively curved locally symmetric Riemannian manifolds  $X$  in a finite cover  $\tilde{X}$  of  $X$  as an extension of the prime geodesic theorem. Especially, when the fundamental group of  $X$  is  $SL_2(\mathbb{Z})$  and that of  $\tilde{X}$  is a congruence subgroup of  $SL_2(\mathbb{Z})$ , we have explicitly determined the type of splitting for each geodesic, and calculate the splitting densities for every types. Applying the results in [HW] to the formula in [VZ] about the relation between the Selberg zeta functions for  $X$  and  $\tilde{X}$ , we can obtain an expression of the Selberg zeta function for  $\tilde{X}$  as a product over prime geodesic of  $X$ . By taking the quotients of such expressions of two Selberg zeta functions, we find a formula of the zeta function given by the Euler product over prime geodesics with a certain type of splitting. From this formula, we have obtained the analytic continuation to  $\{\text{Res} > 0\}$  of such a partial Selberg zeta function (see [HW]).

On the other hand, in [Ku], it was studied the zeta functions defined by the Euler products over prime numbers  $p$  satisfying  $(d/p) = 1$  (or  $= -1$ ) for a fixed square free integer  $d$ . In fact, it was shown that these zeta functions are analytically continued to  $\{\text{Res} > 0\}$  and have natural boundaries on  $\text{Res} = 0$ .

The aim of the present paper is to generalize zeta functions defined by partial Euler products and to study their analytic properties. We first get in Theorem 2.1 the analytic continuations in  $\{\text{Res} > 0\}$  by extending the idea used in [Ku]. Furthermore,

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in Theorem 2.2, we state the sufficient condition of the distributions of non-trivial zeros for the partial zeta functions having natural boundaries on  $\text{Res} = 0$ . As examples, we treat the cases of the Dedekind zeta functions, the Selberg zeta functions and the Ihara zeta functions of graphs. Actually we show that the partial zeta functions of the Dedekind zeta functions for abelian extensions of  $\mathbb{Q}$ , the Selberg zeta functions for congruence subgroups of  $SL_2(\mathbb{Z})$  and the Ihara zeta functions for finite regular graphs have natural boundaries on  $\text{Res} = 0$ .

## 2 Notations and main results

Let  $P$  be an infinite countable set and  $N : P \rightarrow \mathbb{R}_{>1}$  a map such that  $\sum_{p \in P} N(p)^{-d} < \infty$  for some  $d > 0$ . Put  $d_P := \inf\{d > 0 \mid \sum_{p \in P} N(p)^{-d} < \infty\}$  and assume that  $d_P > 0$ . For convenience, we normalize  $N$  by  $d_P = 1$ . We define the zeta function of  $P$  by

$$\zeta_P(s) := \prod_{p \in P} (1 - N(p)^{-s})^{-1} \quad \text{Res} > 1,$$

and assume that (i)  $\zeta_P(s)$  is non-zero holomorphic in  $\{\text{Res} > 1\}$  and has a simple pole at  $s = 1$ , and (ii)  $\zeta_P(s)$  has an analytic continuation to the whole complex plane  $\mathbb{C}$  as a meromorphic function.

Let  $G$  be a finite group and  $\hat{G}$  the set of the finite dimensional irreducible unitary representations of  $G$ . For a map  $\varphi : P \rightarrow \text{Conj}(G)$  and  $\rho \in \hat{G}$ , we define the  $L$ -functions by

$$L_P^{(G)}(s, \rho) = L_P(s, \rho) := \prod_{p \in P} \det(1 - \rho(\varphi(p))N(p)^{-s})^{-1} \quad \text{Res} > 1,$$

and assume that, if  $\rho \neq 1$ , then  $L_P(s, \rho)$  satisfies that (i')  $L_P(s, \rho)$  is non-zero holomorphic in  $\{\text{Res} \geq 1\}$  and satisfies the same condition (ii) for  $\zeta_P(s)$ .

Put  $P_n(G) = P_n := \{p \in P \mid \text{ord}_G(\varphi(p)) = n\}$ , where  $\text{ord}_G(\varphi(p))$  is the order of  $\varphi(p)$  in  $G$ . In the present paper, we study analytic properties of the following zeta function.

$$\zeta_{P_n}(s) := \prod_{p \in P_n} (1 - N(p)^{-s})^{-1} \quad \text{Res} > 1.$$

For simplicity we treat the case where  $G$  is a cyclic group of prime order  $q$ . First we get the following result.

**Theorem 2.1.** *The function  $\zeta_{P_q}(s)$  satisfies the following functional equations.*

$$\frac{(\zeta_{P_q}(s))^q}{\zeta_{P_q}(qs)} = \frac{(\zeta_P(s))^q}{Z_P(s)}, \quad (2.1)$$

where  $Z_P^{(G)}(s) = Z_P(s) := \prod_{\rho \in \hat{G}} L_P(s, \rho)^{\dim \rho}$ . Furthermore, for any  $r \geq 1$ , the function  $(\zeta_{P_q}(s))^{q^r}$  is analytically continued to  $\{\text{Res} > 1/q^r\}$  as a meromorphic function and has infinitely many singular points near  $s = 0$ .

*Proof.* By elementary calculations, we obtain

$$\begin{aligned} Z_P(s) &= \prod_{n \mid \#G} \prod_{\rho \in \hat{G}} \prod_{p \in P_n} \det(1 - \rho(\varphi(p))N(p)^{-s})^{-\dim \rho} \\ &= \prod_{n \mid \#G} \prod_{p \in P_n} (1 - N(p)^{-ns})^{-\#G/n} \\ &= \prod_{n \mid \#G} (\zeta_{P_n}(ns))^{\#G/n}. \end{aligned}$$

Then, when  $G$  is a cyclic group of order  $q$ , we have

$$Z_P(s) = (\zeta_{P_1}(s))^q \zeta_{P_q}(qs).$$

Since  $\zeta_P(s) = \zeta_{P_1}(s)\zeta_{P_q}(s)$ , the equation (2.1) follows immediately.

Now, for convenience, we rewrite the equation (2.1) as follows.

$$\frac{f(s)^q}{f(qs)} = g(s). \quad (2.2)$$

From (2.2), we recursively obtain the following formula.

$$f(s)^{q^r} = f(q^r s) \prod_{i=0}^{r-1} g(q^i s)^{q^{r-i-1}}. \quad (2.3)$$

Since  $g(s)$  is meromorphic and  $f(q^r s)$  is non-zero holomorphic in  $\{\text{Res} > 1/q^r\}$ , we see that  $f(s)^{q^r}$  is meromorphic in  $\{\text{Res} > 1/q^r\}$ .

Since  $g(s) = \zeta_P(s)^q / Z_P(s)$ , the function  $g(s)$  is non-zero holomorphic in  $\{\text{Res} > 1\}$  and has a pole at  $s = 1$  of order  $q - 1$ . Then, according to (2.3), we see that  $f(s)$  has branch points at  $s = 1/q^i$  for  $i = 0, 1, \dots$ . This completes the proof of Theorem 2.1.  $\square$

The theorem above gives the analytic continuation to  $\{\text{Res} > 0\}$  of  $\zeta_{P_q}(s)$ . Next we study analytic properties in  $\{\text{Res} \leq 0\}$  of  $\zeta_{P_q}(s)$ .

Let  $\Lambda$  be the set of singular points in  $\{0 < \text{Res} < 1, \text{Im}s > 0\}$  of  $g(s)$ , and  $m(\sigma)$  is the order of  $\sigma \in \Lambda$  (when  $\sigma$  is a pole,  $m(\sigma)$  is a negative value). Denote by  $\Omega := \{q^{-k}\sigma \mid \sigma \in \Lambda, k \geq 0\}$ . According to (2.3), the set of singular points in  $\{0 < \text{Res} < 1\}$  of  $f(s)$  is a subset of  $\Omega$ . For  $\sigma \in \Lambda$ , we denote by  $[\sigma]$  the subset of  $\Lambda$  which consists of elements  $\sigma' = q^k\sigma \in \Lambda$  for some  $k \in \mathbb{Z}$ . We also denote by  $M_q(\sigma) := \sum_{\sigma' \in [\sigma]} q^{-k}m(\sigma')$  and  $\Lambda_q := \{[\sigma] \subset \Lambda \mid M_q(\sigma) \neq 0\}$ . Under such notations, we obtain the following result.

**Theorem 2.2.** *Number the elements of  $\Lambda_q$  by  $\sigma_0, \sigma_1, \sigma_2, \dots$  such that  $\sigma_i \neq \sigma_j$  for  $i \neq j$  and  $0 < \beta_i \leq \beta_j$  for  $i < j$ , where  $\beta_i := \text{Im}\sigma_i$ . If  $\beta_j \rightarrow \infty$  and  $(\beta_j)^{1/j} \rightarrow 1$  as  $j \rightarrow \infty$ , then the partial zeta function  $\zeta_{P_q}(s)$  has a natural boundary on  $\text{Res} = 0$ .*

*Proof.* According to (2.3), we see that the singular points of  $f(s)$  near the line  $\text{Res} = 0$  consist in

$$\Omega_q := \{q^{-k}\sigma \mid \sigma \in \Lambda_q, k \geq 0\}.$$

We now assume that  $f(s)$  does not have a natural boundary on  $\text{Res} = 0$ , namely there exist constants  $T_1, T_2 > 0$  ( $T_2 > T_1$ ) such that  $\beta_j q^{-k} < T_2$  or  $\beta_j q^{-k} > T_1$  for any  $j, k \geq 0$ . Put  $j(T) := \{j \geq 0 \mid \beta_j \leq T \leq \beta_{j+1}\}$  for  $T > 0$ , and  $J(k) := j(T_1 q^k)$  for a fixed  $k \geq 0$ . By the assumption, we have  $q^{-k}\beta_{J(k)+1} > T_2$ . It is easy to see that  $j(T)$  and  $J(k)$  are non-decreasing functions of  $T$  and  $k$  respectively.

We now estimate  $\beta_{J(k)+1} - \beta_{J(k)}$  for sufficiently large  $k > 0$ . By the definition of  $J(k)$  and the assumption, we have

$$q^{-k}(\beta_{J(k)+1} - \beta_{J(k)}) > T_2 - T_1 > 0. \quad (2.4)$$

On the other hand, since  $(\beta_j)^{1/j} \rightarrow 1$ , we have

$$\beta_{j+1} - \beta_j = o(\beta_j) \quad \text{as } j \rightarrow \infty.$$

Then we obtain

$$q^{-k}(\beta_{J(k)+1} - \beta_{J(k)}) = q^{-k}o(\beta_{J(k)}) < q^{-k}o(T_1 q^k) = o(1) \quad \text{as } k \rightarrow \infty. \quad (2.5)$$

The estimates (2.4) and (2.5) contradict to each other. Then the assumption is false and, therefore, Theorem 2.2 holds.  $\square$

**Remark 2.3.** *It is not difficult to obtain results such as Theorem 2.1 and 2.2 for the case that  $G$  is not necessarily a cyclic group of prime order. For example, when  $\#G = q_1 q_2$  for distinct primes  $q_1$  and  $q_2$ , we can obtain*

$$\frac{\zeta_{P_{q_1 q_2}}(q_1 q_2 s) \zeta_{P_{q_1 q_2}}(s)^{q_1 q_2}}{\zeta_{P_{q_1 q_2}}(q_1 s)^{q_2} \zeta_{P_{q_1 q_2}}(q_2 s)^{q_1}} = \frac{Z_P^{(G)}(s) \zeta_P(s)^{q_1 q_2}}{Z_P^{(H_1)}(s)^{q_2} Z_P^{(H_2)}(s)^{q_1}}, \quad (2.6)$$

where  $H_1$  and  $H_2$  are the subgroups of  $G$  whose orders are  $q_1$  and  $q_2$  respectively. Then, putting

$$f(s) := \frac{\zeta_{P_{q_1 q_2}}(s)^{q_2}}{\zeta_{P_{q_1 q_2}}(q_2 s)}, \quad g(s) := \text{RHS of (2.6)},$$

we have

$$\frac{f(s)^{q_1}}{f(q_1 s)} = g(s).$$

from (2.6). Hence we can obtain the results similar to Theorem 2.1 and 2.2 recursively.

### 3 Examples

#### 3.1 Dedekind zeta functions

Let  $k$  be an algebraic number field over  $\mathbb{Q}$  such that  $[k : \mathbb{Q}] < \infty$ , and  $K$  a finite Galois extension of  $k$ . When we put  $P$  the set of prime ideals of  $k$  unramified in  $K$  and  $N$  the norm of  $\mathfrak{p} \in P$  in  $k$ , we see that  $\zeta_P(s)$  is essentially the Dedekind zeta function

$$\zeta_P(s) = \prod_{\substack{\mathfrak{p} \in \text{Prim}(k) \\ \mathfrak{p} \nmid D}} (1 - N_k(\mathfrak{p})^{-s})^{-1} = \zeta_k(s) \prod_{\substack{\mathfrak{p} \in \text{Prim}(k) \\ \mathfrak{p} \mid D}} (1 - N_k(\mathfrak{p})^{-s}),$$

where  $\text{Prim}(k)$  the set of the prime ideals of  $k$  and  $D$  is the relative discriminant of  $K$  over  $k$ . We also put  $G := \text{Gal}(K/k)$  and  $\varphi$  the Frobenius automorphism. It is well-known that each  $L$ -function satisfies the properties (i') and (ii). Due to the Artin factorization formula, we have

$$Z_P(s) = \prod_{\substack{\mathfrak{p} \in \text{Prim}(K) \\ \mathfrak{p} \nmid D}} (1 - N_K(\mathfrak{p})^{-s})^{-1} = \zeta_K(s) \prod_{\substack{\mathfrak{p} \in \text{Prim}(K) \\ \mathfrak{p} \mid D}} (1 - N_K(\mathfrak{p})^{-s}).$$

For such zeta functions, we obtain the following results.

**Claim 3.1.** *Assume that  $G$  is a cyclic group of prime order  $q$ . Then the partial zeta function  $\zeta_{P_q}(s)$  is analytically continued to  $\{\text{Re } s > 0\}$ . Furthermore, when both  $k$  and  $K$  are abelian extensions of  $\mathbb{Q}$ ,  $\zeta_{P_q}(s)$  has a natural boundary on  $\text{Re } s = 0$ .*

The analytic continuation of  $\zeta_{P_q}(s)$  in  $\{\text{Re } s > 0\}$  is easily obtained by Theorem 2.1. For proving  $\zeta_{P_q}(s)$  has a natural boundary, we prepare the following lemmas.

**Lemma 3.1.** *Let  $\chi_i^{(1)}$  and  $\chi_j^{(2)}$  ( $1 \leq i, j \leq m$ ) be Dirichlet characters respectively modulo  $q_i^{(1)}$  and  $q_j^{(2)}$  such that  $\chi_i^{(1)} \neq \chi_j^{(2)}$ . Denote by*

$$L_1(s) = \prod_{i=1}^m L(s, \chi_i^{(1)}), \quad L_2(s) = \prod_{i=1}^m L(s, \chi_i^{(2)}).$$

Then, for sufficiently large  $T > 0$ , we have

$$\sum_{\substack{0 < \text{Re } \sigma < 1 \\ 0 < \text{Im } \sigma < T \\ L_1(\sigma)L_2(\sigma)=0}} |m_1(\sigma) - m_2(\sigma)| > C \frac{T}{2\pi} \log T,$$

where  $C > 0$  is a constant and  $m_j(\sigma)$  is the multiplicity of  $\sigma$  as a zero of  $L_j(s)$ .

*Proof.* The lemma above for  $m = 1$  has been proved in [Fu]. By applying Bombieri-Perelli's result [BP], we can easily prove that Lemma 3.1 holds for general  $m \geq 1$ .  $\square$

**Lemma 3.2.** *([Co1], [Co2] and [Ba]) Let  $L(s)$  be a Dirichlet  $L$ -function. Then, for  $M \geq 1$  and sufficient large  $T > 0$ , we have*

$$\sum_{\substack{\text{Re } \sigma = 1/2 \\ 0 < \text{Im } \sigma < T \\ L(\sigma) = 0 \\ m(\sigma) \leq M}} m(\sigma) > C_M \frac{T}{2\pi} \log T,$$

where  $C_M > 0$  is a constant which satisfies  $C_M = 1 - O(M^{-2})$  as  $M \rightarrow \infty$ .

**Lemma 3.3.** *([Mo]) Let  $L(s)$  be a Dirichlet  $L$ -function. Then, for  $0 < \alpha < 1/2$ , we have*

$$\sum_{\substack{0 < \text{Re } \sigma < \alpha \\ 0 < \text{Im } \sigma < T \\ L(\sigma) = 0}} m(\sigma) \ll T^{5/2\alpha + \epsilon}.$$

**Proof of Claim 3.1.** Let  $n := [K : \mathbb{Q}]$ . When  $k$  and  $K$  are abelian extensions of  $\mathbb{Q}$ , the Dedekind zeta functions  $\zeta_k(s)$  and  $\zeta_K(s)$  are expressed by products of Dirichlet  $L$ -functions. Then  $g(s)$  is written as

$$g(s) = \left( \prod_{i=1}^n L(s, \chi_i^{(1)}) \right) / \left( \prod_{i=1}^n L(s, \chi_i^{(2)}) \right).$$

From Lemma 3.1, we see that there exists a constant  $C > 0$  such that

$$\sum_{\substack{0 < \operatorname{Re} \sigma < 1 \\ 0 < \operatorname{Im} \sigma < T \\ g(\sigma) = 0}} m(\sigma) > C \frac{T}{2\pi} \log T.$$

Due to Lemma 3.2, we have

$$\sum_{\substack{\operatorname{Re} \sigma = 1/2 \\ 0 < \operatorname{Im} \sigma < T \\ g(\sigma) = 0 \\ m(\sigma) \leq M}} m(\sigma) > \tilde{C}_M \frac{T}{2\pi} \log T,$$

where  $\tilde{C}_M = m - O(M^{-2})$  as  $M \rightarrow \infty$ . Then, taking  $M$  such that  $C + \tilde{C}_M > m$ , we obtain

$$I(T) := \sum_{\substack{\operatorname{Re} \sigma = 1/2 \\ 0 < \operatorname{Im} \sigma < T \\ g(\sigma) = 0}} 1 > \frac{C + \tilde{C}_M - m}{M} \frac{T}{2\pi} \log T.$$

Furthermore, from Lemma 3.3, we have

$$J_\alpha(T) := \sum_{\substack{0 < \operatorname{Re} \sigma < \alpha \\ 0 < \operatorname{Im} \sigma < T \\ g(\sigma)^{-1} = 0}} m(\sigma) \ll T^{5/2\alpha + \epsilon}.$$

Then we have

$$\Omega_q(T) := \#\{\sigma \in \Omega_q \mid 0 < \operatorname{Im} \sigma < T\} > I(T) - \sum_{l \geq 1} J_{q^{-l-1}}(q^l T) = O(T \log T).$$

This implies that the conditions in Theorem 2.2 are satisfied and, therefore, the partial zeta function has a natural boundary on  $\operatorname{Re} s = 0$ .  $\square$

### 3.2 Selberg (Ruelle) zeta functions

Let  $\mathbb{H}$  be the upper half plane and  $\Gamma$  a discrete subgroup of  $SL_2(\mathbb{R})$  such that the volume of  $X_\Gamma = \Gamma \backslash \mathbb{H}$  is finite. Put  $P = \text{Prim}(\Gamma)$  the set of the primitive conjugacy classes of  $\Gamma$  and  $N(p)$  the square of the larger eigenvalue of  $p \in \text{Prim}(\Gamma)$ . Then it is known that  $\zeta_P(s)$  coincides the Selberg (Ruelle) zeta function

$$\zeta_\Gamma(s) := \prod_{p \in \text{Prim}(\Gamma)} (1 - N(p)^{-s})^{-1} \quad \text{Res} > 1$$

which satisfies the conditions (i) and (ii) (see, e.g. [He]).

Fix  $\Gamma'$  a normal subgroup of  $\Gamma$  of finite index. Put  $G = \Gamma/\Gamma'$  and  $\varphi$  a natural projection from  $\text{Conj}(\Gamma)$  to  $\text{Conj}(G)$ . It is known that  $L$ -functions satisfies (i') and (ii). According to [VZ], we have

$$\zeta_{\Gamma'}(s) := \prod_{p \in \text{Prim}(\Gamma')} (1 - N(p)^{-s})^{-1} \quad \text{Res} > 1.$$

For the Selberg zeta function, we obtain the following result.

**Claim 3.2.** *Assume that  $G$  is a cyclic group of odd prime order  $q$ . Then  $\zeta_{P_q}(s)$  is analytically continued to  $\{\text{Res} > 0\}$ . Furthermore, when both  $\Gamma$  and  $\Gamma'$  are congruence subgroups of  $SL_2(\mathbb{Z})$ , the partial zeta function  $\zeta_{P_q}(s)$  has a natural boundary on  $\text{Res} = 0$ .*

*Proof.* Similar to the previous section, the analytic continuation of  $\zeta_{P_q}(s)$  is obtained by Theorem 2.1.

When  $\Gamma$  is a congruence subgroup of  $SL_2(\mathbb{Z})$ , due to the determinant expression of the Selberg zeta function (see [Hu] and [Ko]), we have

$$\Lambda = \{1/2 + ir_j \mid \lambda_j > 1/4\} \cup \bigcup_{l=1}^{h_\Gamma} \{\sigma \mid L(2\sigma, \chi_l) = 0, 0 < \text{Re}\sigma < 1/2, \text{Im}(\sigma) > 0\},$$

where  $\lambda_j = 1/4 + r_j^2$  is the  $j$ -th eigenvalue of the Laplacian on  $X_\Gamma$  with the multiplicity  $m_j$ ,  $h_\Gamma$  is the number of cusps for  $\Gamma$  and  $\chi_1, \dots, \chi_{h_\Gamma}$  are Dirichlet characters determined by  $\Gamma$ . Then, similar to the case of the previous section, we can easily check that the conditions in Theorem 2.2 are satisfied for  $q > 2$ . Therefore we obtain Claim 3.2.  $\square$

**Remark 3.4.** *When  $X_\Gamma$  and  $X'_\Gamma$  are compact Riemann surfaces, it is known that  $\Lambda = \{1/2 + ir_j\}$*

$$\sum_{|r_j| < T} m_j \sim \frac{\text{vol}(X_\Gamma)}{4\pi} T^2, \quad m_j \ll \frac{r_j}{\log r_j}.$$



If it would be known that there exists a constant  $\delta > 0$  such that

$$\sum_{\substack{\sigma=1/2+i\beta \\ 0<\beta<T \\ \{f_1(\sigma)f_2(\sigma)\}^{-1}=0}} |m_1(\sigma) - m_2(\sigma)| \gg T^{1+\delta} \quad (3.1)$$

for  $f_1(s) := \zeta_\Gamma(s)^q$ ,  $f_2(s) := \zeta_{\Gamma'}(s)$  and the orders  $m_1(\sigma), m_2(\sigma)$  respectively of the singular points of  $f_1(s), f_2(s)$ , then we could prove that the partial zeta function has a natural boundary on  $\text{Res} = 0$ . However, no results such like Lemma 3.1 or (3.1) have been obtained for the Selberg zeta functions. Hence we cannot presently conclude that whether the partial Selberg zeta function for the compact case has a natural boundary on  $\text{Res} = 0$ .

### 3.3 Ihara zeta functions

Let  $X$  be a finite connected  $(q+1)$ -regular graph with  $n$  vertices and  $Y$  a finite connected  $(q+1)$ -regular unramified covering graph of  $X$ . Denote  $\Gamma_X$  and  $\Gamma_Y$  by the fundamental groups of  $X$  and  $Y$  respectively, and assume that  $\Gamma_Y$  is a normal subgroup of  $\Gamma_X$ . Put  $P$  the set of the equivalence classes of primitive closed backtrackless tail-less cycles. Then the Ihara zeta function of  $X$  are defined by

$$\zeta_X(u) := \prod_{p \in P} (1 - u^{\nu(p)}) \quad |u| < 1,$$

where  $\nu(p)$  is the number of the edges in the cycle  $p$ . It is known that  $\zeta_X(s)$  has the following determinant expression (see, e.g. [Ih]).

$$\zeta_X(u)^{-1} = (1 - u^2)^{n(q-1)/2} \det(\text{Id} - Au + qu^2 \text{Id}), \quad (3.2)$$

where  $A$  is the adjacency matrix of  $X$ . From the formula (3.2), we see that  $\zeta_X(u)$  has a finite number of poles in  $\{q^{-1} \leq |u| \leq 1\}$ . Note that the pole at  $u = q^{-1}$  is simple. For  $N(p) := q^{\nu(p)}$ , we have  $\zeta_P(s) = \zeta_X(q^{-s})$ . According to (3.2), we see that  $\zeta_P(s)$  satisfies the property (i) and (ii).

Take  $G = \Gamma_X / \Gamma_Y$ . It is known that the  $L$ -functions satisfy the conditions (i'), (ii) and  $Z_P^{(G)}(s) = \zeta_Y(q^{-s})$  (see [ST]). Due to (3.2), we see that the singular points of  $\zeta_P(s)$  and  $Z_P(s)$  are periodically distributed. Then we can easily prove the following results by using Theorem 2.1 and 2.2.

**Claim 3.3.** *Assume that  $G$  is a cyclic group of odd prime order  $q$ . Then  $\zeta_{P_q}(s)$  is analytically continued to  $\{\text{Res} > 0\}$  has a natural boundary on  $\text{Res} = 0$ .  $\square$*

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## References

- [Ba] P. J. Bauer, *Zeros of Dirichlet  $L$ -series on the critical line*, Acta. Arith. **93** (2000), 37–52.
- [BP] E. Bombieri and A. Perelli, *Distinct zeros of  $L$ -functions*, Acta. Arith. **83** (1998), 271–281.
- [Co1] B. Conrey, *More than two-fifth of the zeros of the Riemann zeta-function are on the critical line*, J. Reine Angew. Math. **399** (1989) 1–26.
- [Co2] B. Conrey, *Zeros of derivatives of Riemann's  $\Xi$ -function on the critical line. II*, J. Number Theory **17** (1983) 71–75.
- [Fu] A. Fujii, *On the zeros of Dirichlet  $L$ -functions (V)*, Acta Arith. **28** (1976) 395–403.
- [HW] Y. Hashimoto and M. Wakayama, *Splitting density for lifting about discrete groups*, math.NT/0501284.
- [He] D. Hejhal, *The Selberg trace formula of  $PSL(2, \mathbb{R})$  I, II*, Springer Lec. Notes in Math. **548**, **1001** Springer-Verlag, (1976, 1983).
- [Hu] M. N. Huxley, *Scattering matrices for congruence subgroups*, Modular forms (Ellis Horwood Ser. Math. Appl. 1984), 141–156.
- [Ih] Y. Ihara, *On discrete subgroups of the two by two projective linear group over  $\mathfrak{p}$ -adic fields*, J. Math. Soc. Japan **18** (1966), 219–235.
- [Iw] H. Iwaniec, *Spectral Methods of Automorphic Forms*, Graduate Studies in Mathematics, **53**, 2nd edition, American Mathematical Society, (2002).
- [Ko] S. Koyama, *Determinant expressions of Selberg zeta functions I*, Trans. Amer. Math. Soc. **324** (1991), 149–168.
- [Ku] N. Kurokawa, *On certain Euler products*, Acta Arith. **48** (1987), 49–52.
- [Mo] H. L. Montgomery, *Zeros of  $L$ -functions*, Invent. Math. **8** (1969), 346–354.
- [ST] H. M. Stark and A. A. Terras, *Zeta functions of finite graphs and coverings, II*, Adv. in Math. **154** (2000), 132–195.
- [Su] T. Sunada,  *$L$ -functions in geometry and some applications*, Curvature and topology of Riemannian manifolds (Katata, 1985), 266–284, Lecture Notes in Math., **1201**, Springer, Berlin (1986).

- [Ti] E. C. Titchmarsh, *The theory of the Riemann zeta function*, 2nd ed., Clarendon Press, 1988.
- [VZ] A. B. Venkov and P. G. Zograf, *Analogues of Artin's factorization formulas in the spectral theory of automorphic functions associated with induced representations of Fuchsian groups*, *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), 1150–1158, 1343 (Russian), *Math. USSR-Izv.* **21** (1983), 435–443 (English translation).

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